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GRIOLI-TYPE PRECESSION OF A HEAVY RIGID BODY IN A LIQUID†

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Grioli-type precession relative to an inclined axis is considered in the problem of the motion of a heavy rigid body in a liquid, described by the Kirchhoff–Clebsch equations with Chaplygin's condition. Conditions for the existence of a solution, corresponding to this class of motions in the case when the body's axes of spin and precession are not perpendicular to one another, are obtained.

Let us assume that a free rigid body, bounded by a simply-connected surface, is moving under the action of a uniform gravitational force field in an unlimited volume of ideal incompressible liquid in irrotational motion and at rest at infinity. In the body there is a symmetrically shaped flywheel, rotating at a constant angular velocity about an axis rigidly attached to the carrier-body. We will also assume that the weight of the liquid displaced by the body is equal to the weight of the body plus the flywheel.

Referring to a system of coordinates $Ox_1x_2x_3$ rigidly attached to the body, we introduce an impulsive force $\mathbf{R} = (R_1, R_2, R_3)$ – the momentum vector of the "gyrostat plus liquid" system, and an impulsive momentum $\mathbf{P} = (P_1, P_2, P_3)$ – the angular momentum vector of the system about the point O. Then the kinetic energy of the system is [1–3]

$$T = \frac{1}{2}\sum_{i,j=1}^{n}(a_{ij}P_iP_j + b_{ij}R_iR_j + 2c_{ij}P_iR_j), \quad a_{ij} = a_{ji}, \quad b_{ij} = b_{ji} \tag{1}$$

where a_{ij}, b_{ij} and c_{ij} are constants, which depend on the characteristics of the "gyrostat plus liquid" system. Let λ_1, λ_2 and λ_3 denote the projections of the gyrostatic moment of the flywheel onto the x_i axes, and let e_1, e_2 and e_3 be constants proportional to the projections onto the x_i axes of a radius-vector drawn from the centre of gravity of the volume bounded by the external surface of the body to the centre of mass of the gyrostat. The components of the vectors of translational velocity $\mathbf{u}$ and instantaneous angular velocity $\mathbf{\Omega}$ are defined by the relations

$$u_i = \partial T/\partial R_i, \qquad \Omega_i = \partial T/\partial P_i, \qquad i = 1, 2, 3 \tag{2}$$

Let us assume that the impulsive force $\mathbf{R}$ is directed along the upward vertical. According to well-known results [1–4], the equations of motion of the "gyrostat plus liquid" system may be written in the form

$$dP_1/dt + \Omega_2(P_3+\lambda_3) - \Omega_3(P_2+\lambda_2) + u_2R_3 - u_3R_2 = e_2R_3 - e_3R_2 \qquad (123) \tag{3}$$

$$dR_1/dt + \Omega_2R_3 - \Omega_3R_2 = 0 \qquad (123) \tag{4}$$

Thus, Eqs (3) and (4) describe inertial motion in an unlimited liquid of a rigid body bounded by a simply connected surface [4]. The symbol (123) means that the other equations are obtained by cyclic permutation of the indices 1, 2, 3.

Let γ be a fixed unit vector in the immovable space, and let $\mathbf{a}$ be a unit vector fixed in the body ($\dot{\mathbf{a}} = \mathbf{0}$).

The motion of the body is said to be precessional [5] if the angle θ_0 between the vectors $\mathbf{a}$ and γ remains constant throughout the motion. By definition, the condition for the motion to be precessional may be characterized by the following invariant relation

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$$\mathbf{a}\cdot\boldsymbol{\gamma}=a_0,\qquad a_0=\cos\theta_0$$

Let $\boldsymbol{\nu}$ be a unit vector characterizing the fixed direction in space of the impulsive force $\mathbf{R} = H\boldsymbol{\nu}$ ($\mathbf{R}^2 = H^2 =$ const). Without loss of generality, we will assume that the vector $\mathbf{a}$ points in the direction of the x_3 axis. We attach an immovable system of coordinates to the vector $\boldsymbol{\gamma}$, introducing the Euler angles θ_0, φ and ψ, so that

$$\begin{aligned}&\boldsymbol{\gamma}=(a_0'\sin\varphi, a_0'\cos\varphi, a_0),\quad a_0'=\sin\theta_0,\quad c_0=\cos\sigma_0,\quad b_0=\sin\sigma_0\\&\boldsymbol{\nu}=(c_0+a_0'b_0'\sin\psi)\boldsymbol{\gamma}-b_0'\mathbf{a}\sin\psi-b_0'(\boldsymbol{\gamma}\times\mathbf{a})\cos\psi,\quad b_0'=b_0/a_0'\end{aligned}\tag{5}$$

where σ_0 is the angle between the vectors ν and γ.

Then the angular velocity of the body can be expressed in the form

$$\boldsymbol{\Omega}=\dot{\varphi}\mathbf{a}+\dot{\psi}\boldsymbol{\gamma}\tag{6}$$

This relation is the kinematic condition for the motion of a rigid body with a fixed point to be precessional [5].

Suppose the motion of the gyrostat is regular Grioli-type precession about an inclined axis [6]. Then the condition $\dot{\varphi} = \dot{\psi}$ is satisfied. A suitable choice of the fixed system of coordinates will ensure that $\varphi = \psi = mt$, where m is a constant. Then, taking relations (5) and (6) into consideration, we have

$$\begin{aligned}&\Omega_1=ma_0'\sin\varphi,\quad \Omega_2=ma_0'\cos\varphi,\quad \Omega_3=m(1+a_0)\\&R_1=H(a_0'c_0\sin\varphi-\frac{b_0}{2}(1-a_0)-\frac{b_0}{2}(1+a_0)\cos2\varphi)\\&R_2=H(a_0'c_0\cos\varphi+\frac{b_0}{2}(1+a_0)\sin2\varphi),\quad R_3=H(a_0c_0-b_0a_0'\sin\varphi)\end{aligned}\tag{7}$$

It can be verified that Eqs (4) are satisfied identically by relations (7).

On the basis of relations (2), we express the vectors $\mathbf{P}$ and $\mathbf{u}$ in this form

$$\begin{aligned}&\mathbf{P}=\mathbf{A}\boldsymbol{\Omega}-\mathbf{C}\mathbf{R},\quad \mathbf{u}=\mathbf{C}^{\tau}\boldsymbol{\Omega}+\mathbf{B}\mathbf{R}\\&\mathbf{A}=\| A_{ij}\|_1^3=a^{-1},\quad \mathbf{B}=\| B_{ij}\|_1^3=b-c^{\tau}a^{-1}c,\quad \mathbf{C}=\| C_{ij}\|_1^3=a^{-1}c\\&a=\| a_{ij}\|_1^3,\quad b=\| b_{ij}\|_1^3,\quad c=\| c_{ij}\|_1^3\end{aligned}\tag{8}$$

Substituting expressions (7) and (8) into the first of Eqs (3) and stipulating that the resulting equality should be an identity in the variable φ, we find the conditions

$$\begin{aligned}&B_{22}=B_{11},\quad B_{12}=0,\quad C_{21}=-C_{12}+\frac{Hb_0}{2m}B_{23},\quad C_{22}=C_{11}-\frac{Hb_0}{2m}B_{13}\\&m^2(1-a_0)(A_{22}-A_{11})+H^2b_0c_0a_0B_{13}-Hb_0e_1-mHb_0(1-a_0)(C_{13}+C_{31})=0\\&Hc_0(1+2a_0)B_{23}-e_2+m(1+a_0)(C_{23}+C_{32})=0\\&H^2b_0c_0a_0B_{23}-2m^2(1-a_0)A_{12}-Hb_0e_2-mHb_0(1-a_0)(C_{23}+C_{32})=0\\&\tfrac{1}{2}H^2b_0(c_0^2(1+a_0)+a_0b_0^2)B_{23}-m^2(1-a_0)c_0A_{12}+mb_0\lambda_2+m^2b_0a_0A_{23}=0\\&m^2(1-a_0)c_0(A_{11}-A_{22})-\tfrac{1}{2}H^2b_0^3B_{13}-mb_0\lambda_1-m^2b_0a_0A_{13}+\\&+mHb_0c_0(C_{13}+C_{31})+mHb_0^2C_{33}=0\end{aligned}\tag{9}$$

Taking relation (9) into consideration, we substitute expressions (7) and (8) into the second of Eqs (3) and, stipulating that the resulting equality should be an identity in the variable φ, we find

$$
\begin{aligned}
&B_{13} = 0, \quad B_{23} = 0, \quad A_{12} = 0, \quad m(A_{22} - A_{11}) - Hb_0(C_{13} + C_{31}) = 0, \quad A_{23} = 0 \\
&ma_0A_{13} + Hb_0(1 - a_0)C_{11} + Hb_0a_0C_{33} = 0 \\
&mb_0[a_0A_{22} - (1 + a_0)A_{33}] - b_0\lambda_3 + mc_0(1 + a_0)A_{13} + \\
&+Hb_0c_0[(1 + 2a_0)C_{33} - 2a_0C_{11}] = 0
\end{aligned}
\tag{10}
$$

By (10) we deduce from (9) that

$$
\begin{aligned}
&C_{22} = C_{11}, \quad C_{21} = -C_{12}, \quad C_{23} = -C_{32}, \quad e_1 = e_2 = 0, \quad \lambda_2 = 0 \\
&Ha_0c_0(C_{13} + C_{31}) + Hb_0C_{33} - \lambda_1 - ma_0A_{13} = 0
\end{aligned}
\tag{11}
$$

Using the restrictions obtained on the parameters of problem (10), (11), we substitute expressions (7) and (8) into the third of Eqs (3). The resulting equality will be an identity in the variable φ provided that

$$
\begin{aligned}
&m^2(A_{22} - A_{11}) + H^2b_0^2(B_{11} - B_{33}) = 0 \\
&m^2(1 + a_0)A_{13} + H^2a_0b_0c_0(B_{11} - B_{33}) + Hb_0e_3 = 0
\end{aligned}
\tag{12}
$$

Relationships (10)–(12) can be conveniently expressed as follows:

$$
\begin{aligned}
&A_{12} = A_{23} = 0, \quad B_{22} = B_{11}, \quad B_{12} = B_{13} = B_{23} = 0, \quad \lambda_2 = 0 \\
&e_1 = e_2 = 0, \quad C_{22} = C_{11}, \quad C_{21} = -C_{12}, \quad C_{23} = -C_{32}, \quad \chi_0 = C_{13} + C_{31} \\
&\chi_1 = C_{33} + C_{11} + a_0(C_{33} - C_{11}), \quad \chi_2 = A_{13}\chi_0 + (A_{22} - A_{11})(C_{33} - C_{11}) \\
&B_{33} = B_{11} + \frac{\chi_0^2}{A_{22} - A_{11}}, \quad a_0 = \frac{C_{11}(A_{11} - A_{22})}{\chi_2} \\
&m = \frac{Hb_0\chi_0}{A_{22} - A_{11}}, \quad H = \frac{\lambda_1}{b_0\chi_1 + c_0a_0\chi_0} \\
&\lambda_1 = \frac{e_3(A_{22} - A_{11})^2(b_0\chi_1 + c_0a_0\chi_0)}{\chi_0[b_0((A_{22} - A_{11})(\chi_1 - C_{33}) - A_{13}\chi_0) + c_0a_0\chi_0(A_{22} - A_{11})]} \\
&\frac{b_0}{c_0} = \frac{a_0\lambda_3\chi_0^2 - e_3[(1 + a_0)A_{13}\chi_0 + (C_{33} + 2a_0(C_{33} - C_{11}))(A_{22} - A_{11})]}{e_3\chi_0(a_0A_{22} - (1 + a_0)A_{33}) + (1 + a_0)\lambda_3A_{13}(B_{33} - B_{11})}
\end{aligned}
\tag{13}
$$

If conditions (13) are satisfied, the motion of the body will be Grioli precession about an inclined axis.

Note that in the case considered the body's axes of spin and precession will not be mutually perpendicular. Setting $a_0 = 0$ in (13), we obtain results established by Rubanovskii [7].

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